AN ROINN OIDEACHAIS AGUS EOLAÍOCHTA | DEPARTMENT OF EDUCATION AND SCIENCE

Scéimeanna Marcála	*Scrúduithe Ardteistiméireachta,* 2001
Matamaitic Fheidhmeach	*Ardleibhéal*

Marking Scheme	*Leaving Certificate Examination,* 2001
Applied Mathematics	*Higher Level*

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General Guidelines

1 Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	S(-1)
Blunders	- mathematical errors	B(-3)
Misreading	- if not serious	M(-1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (att 2).

2 Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3 Mark scripts in red unless candidate uses red. If a candidate uses red, mark the script in blue or black.

4 Number the grid on each script 1 to 10 in numerical order, not the order of answering.

5 Scrutinise **all** pages of the answer book.

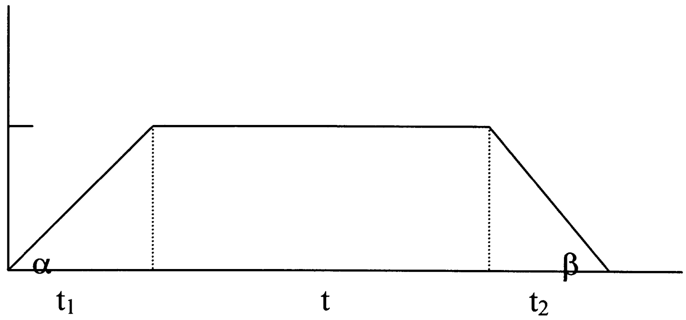
6 The marking scheme shows one correct solution to each question. In many cases there are other equally valid methods.

1. **(a)** Points p and q lie in a straight line, where $|pq| = 1200$ metres.

Starting from rest at p, a train accelerates at 1 m/s^2 until it reaches the speed limit of 20 m/s. It continues at this speed of 20 m/s and then decelerates at 2 m/s^2, coming to rest at q.

Find the time it takes the train to go from p to q.

Find the shortest time it takes the train to go from rest at p to rest at q if there is no speed limit, assuming that the acceleration and deceleration remain unchaged at 1 m/s^2 and 2 m/s^2 , respectively.



$\tan\alpha = \frac{20}{t_1} \Rightarrow 1 = \frac{20}{t_1} \Rightarrow t_1 = 20$ 5

$\tan\beta = \frac{20}{t_2} \Rightarrow 2 = \frac{20}{t_2} \Rightarrow t_2 = 10$ 5

distance = area

$1200 = \frac{1}{2}(20)(20) + (20)t + \frac{1}{2}(10)(20)$

$1200 = 200 + 20t + 100$

$\Rightarrow t = 45$

$\Rightarrow$ time to go from p to q $= 20 + 45 + 10 = 75$ seconds. 5

$\tan\alpha = \frac{v}{t_1} \Rightarrow 1 = \frac{v}{t_1} \Rightarrow t_1 = v$ or $t_2 = \frac{v}{2}$ 5

$\tan\beta = \frac{v}{t_2} \Rightarrow 2 = \frac{v}{t_2} \Rightarrow t_2 = \frac{v}{2}$

distance = area

$1200 = \frac{1}{2}(v)(v) + \frac{1}{2}(\frac{1}{2}v)(v)$

$1200 = \frac{3}{4}v^2$

$\Rightarrow v = 40$

$\Rightarrow$ shortest time to go from p to q $= 40 + 20 = 60$ seconds. 5

25

(b) A particle is projected vertically upwards with an initial velocity of *u* m/s and another particle is projected vertically upwards from the same point and with the same initial velocity *T* seconds later.

Show that the particles

(i) will meet $\left(\frac{T}{2}+\frac{u}{g}\right)$ seconds from the instant of projection of the first particle

(ii) will meet at a height of $\frac{4u^2 - g^2T^2}{8g}$ metres.

$s_1 = u(T + t_1) - \frac{1}{2}g(T + t_1)^2$ or $s_1 = ut_2 - \frac{1}{2}gt_2{}^2$ 5

$s_2 = ut_1 - \frac{1}{2}gt_1{}^2$ or $s_2 = u(t_2 - T) - \frac{1}{2}g(t_2 - T)^2$ 5

$s_1 = s_2$ 5

$\Rightarrow T(u - \frac{1}{2}gT - gt_1) = 0$ or $T(u + \frac{1}{2}gT - gt_2) = 0$

$\Rightarrow t_1 = \frac{u}{g} - \frac{T}{2}$

$\Rightarrow T + t_1 = \frac{u}{g} + \frac{T}{2}$ or $t_2 = \frac{u}{g} + \frac{T}{2}$ 5

height $= s_1 = s_2$

$= u\left\{\frac{u}{g} \pm \frac{T}{2}\right\} - \frac{1}{2}g\left\{\frac{u}{g} \pm \frac{T}{2}\right\}^2$

$= \frac{4u^2 - g^2T^2}{8g}$ 5 25

2. **(a)** Ship B is travelling at $5\sqrt{34}$ km/hr in the direction $\tan^{-1}\frac{6}{7}$ north of east and ship C is travelling at $5\sqrt{5}$ km/hr in the direction $\tan^{-1}7$ north of west.

Show that the speed and direction of ship B relative to ship C is 25 km/hr at $\tan^{-1}\frac{1}{3}$ north of east.

$V_B = 5\sqrt{34}\left(\frac{7}{\sqrt{85}}\right)\vec{i} + 5\sqrt{34}\left(\frac{6}{\sqrt{85}}\right)\vec{j}$ or $7\sqrt{10}\,\vec{i} + 6\sqrt{10}\,\vec{j}$

or $22.14\,\vec{i} + 18.97\,\vec{j}$ 5

$V_C = -5\sqrt{5}\left(\frac{1}{\sqrt{50}}\right)\vec{i} + 5\sqrt{5}\left(\frac{7}{\sqrt{50}}\right)\vec{j}$ or $-0.5\sqrt{10}\,\vec{i} + 3.5\sqrt{10}\,\vec{j}$

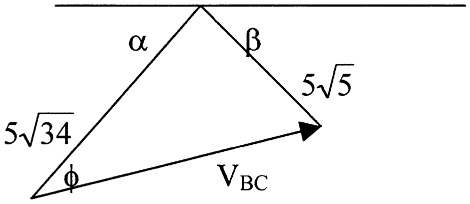
or $-1.58\,\vec{i} + 11.07\,\vec{j}$ 5

$V_{BC} = V_B - V_C$ 5

$= 7.5\sqrt{10}\,\vec{i} + 2.5\sqrt{10}\,\vec{j}$ or $23.72\,\vec{i} + 7.90\,\vec{j}$ 5

magnitude is 25 km/hr

direction is $\tan^{-1}\left(\frac{1}{3}\right)$ or $\tan^{-1}(0.33)$ or 18.4° north of east 5 25



$V_{BC} = V_B - V_C$ or vector diagram 5

$|V_{BC}|^2 = \left(5\sqrt{34}\right)^2 + \left(5\sqrt{5}\right)^2 - 2\left(5\sqrt{34}\right)\left(5\sqrt{5}\right)\cos\{180 - (\alpha + \beta)\}$ 5

$\cos\{180 - (\alpha + \beta)\} = \frac{7}{\sqrt{170}}$ or 0.5369 5

magnitude = 25 km/hr

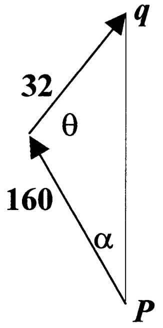
$\{180 - (\alpha + \beta)\} = 57.53°$

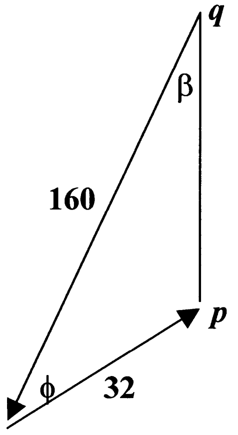
$\sin\phi = 5\sqrt{5}\{\sin 57.53\} \div 25 \quad \Rightarrow \quad \phi = 22.17°$ 5

direction = $\alpha - \phi = 40.60 - 22.17$ or 18.4° north of east 5 25

(b) The speed of an aeroplane in still air is 160 km/hr. It flies in a straight line from p to q and back again. Point q is due north of point p. Throughout the journey there is a wind blowing from the south-west at 32 km/hr. The time for the whole journey is 5 hours.

Find the distance from p to q. Give your answer to the nearest km.





$160\sin\alpha = 32\sin 45$ or $\dfrac{\sin\alpha}{32} = \dfrac{\sin 45}{160}$ 5

$\Rightarrow \sin\alpha = \dfrac{1}{5\sqrt{2}}$ $\{\alpha = 8.13°\ ,\ \theta = 126.87°\}$ 5

$$V_{pq} = \frac{160\sin126.87°}{\sin45°} \text{ or } \frac{32\sin126.87°}{\sin8.13°}$$

or $160\cos8.13° + 32\cos45$

or $128\sqrt{2}$ or 181.02 5

$160\sin\beta = 32\sin45 \Rightarrow \sin\beta = \dfrac{1}{5\sqrt{2}}$ $\{\beta = 8.13°,\ \phi = 36.87°\}$

$$V_{qp} = \frac{160\sin36.87°}{\sin135°} \text{ or } \frac{32\sin36.87°}{\sin8.13°}$$

or $160\cos8.13° - 32\cos45$

or $96\sqrt{2}$ or 135.76 5

$$t_1 + t_2 = 5$$

$$\frac{d}{181.02} + \frac{d}{135.76} = 5$$

$\Rightarrow d = 387.9$ or 388 km 5

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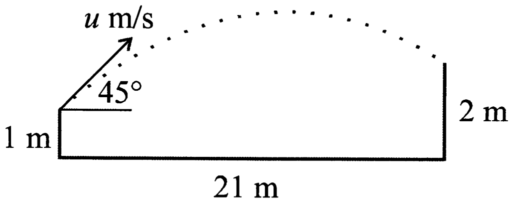
3. **(a)** A player hits a ball with an initial speed of u m/s from a height of 1 m at an angle of 45°to the horizontal ground.
A member of the opposing team, 21 m away, catches the ball at a height of 2 m above the ground.



Find the value of u.

$$r_i = 21 \qquad 5$$

$$u\cos 45.t = 21 \qquad 5$$

$$t = \frac{21}{u\cos 45} \text{ or } \frac{21\sqrt{2}}{u}$$

$$r_j = 1 \qquad 5$$

$$u\sin 45.t - \tfrac{1}{2}gt^2 = 1 \qquad 5$$

$$u\left(\frac{1}{\sqrt{2}}\right)\left(\frac{21\sqrt{2}}{u}\right) - 4.9\left(\frac{21\sqrt{2}}{u}\right)^2 = 1$$

$$21 - \frac{4321.8}{u^2} = 1$$

$$u = 14.7 \qquad 5$$

$$\text{or } \frac{3g}{2} \text{ or } 21\sqrt{\frac{g}{20}}$$

$$\boxed{25}$$

$$u\sin 45.t - \tfrac{1}{2}gt^2 = 1$$

$$21 - \tfrac{1}{2}gt^2 = 1$$

$$\tfrac{1}{2}gt^2 = 20$$

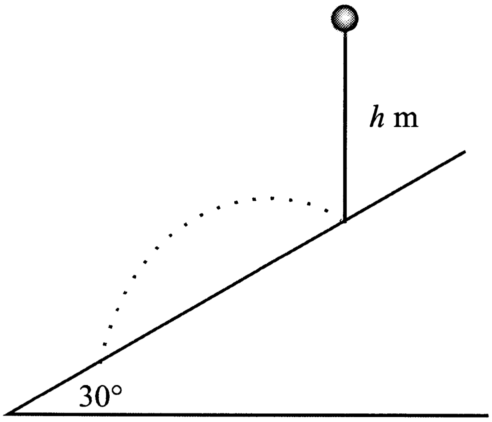
$$t = \sqrt{\frac{40}{g}}$$

$$u = \frac{21\sqrt{2}}{t} = 21\sqrt{2}\sqrt{\frac{g}{40}}$$

$$= 21\sqrt{\frac{g}{20}} = 21(0.7) = 14.7$$

(b) A ball is dropped from a height of h m onto a smooth inclined plane. The ball strikes the plane at p and rebounds.
The plane is inclined at an angle of 30° to the horizontal and the coefficient of restitution between the ball and the plane is $\frac{1}{2}$.

Find how far down the plane from p is the ball's next point of impact.
Express your answer in terms of h,



$$v^2 = u^2 + 2as$$
$$v^2 = 0 + 2gh \quad \Rightarrow \quad v = \sqrt{2gh}$$ 5

rebound velocity $= v\sin 30\,\vec{i} + ev\cos 30\,\vec{j}$ 5

$$= \sqrt{2gh}\sin 30\,\vec{i} + \tfrac{1}{2}\sqrt{2gh}\cos 30\,\vec{j}$$

$$r = \left\{v\sin 30.t + \tfrac{1}{2}g\sin 30.t^2\right\}\vec{i} + \left\{ev\cos 30.t - \tfrac{1}{2}g\cos 30.t^2\right\}\vec{j}$$ 5

$$= \left\{\sqrt{2gh}\sin 30.t + \tfrac{1}{2}g\sin 30.t^2\right\}\vec{i} + \left\{\tfrac{1}{2}\sqrt{2gh}\cos 30.t - \tfrac{1}{2}g\cos 30.t^2\right\}\vec{j}$$

$$r_j = 0 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}} \text{ or } \frac{v}{g}$$ 5

$$\Rightarrow \quad r_i = \left\{\sqrt{2gh}\sin 30.\sqrt{\frac{2h}{g}} + \tfrac{1}{2}g\sin 30.\frac{2h}{g}\right\} = \frac{3\text{h}}{2}$$ 5

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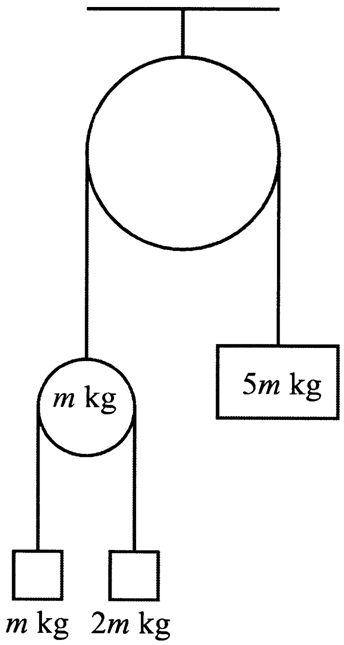
4. A smooth pulley, of mass m kg, is connected by a light inextensible string passing over a smooth light fixed pulley to a particle of mass $5m$ kg.
Two particles of masses m kg and $2m$ kg are connected by a light inextensible string passing over the smooth pulley of mass m kg.



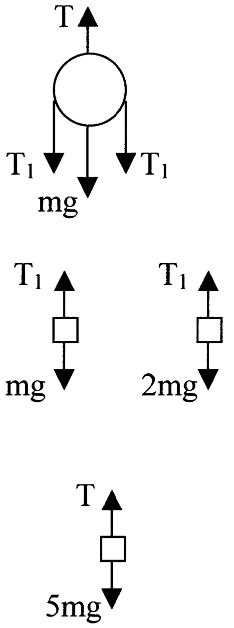
The system is released from rest.

(i) Draw a diagram showing all the forces acting on each particle and on the smooth pulley of mass m kg.

(ii) Find the acceleration of each particle, in terms of g.

(iii) When the particle of mass $2m$ kg has moved down 1 metre relative to the fixed pulley, find how far the particle of mass $5m$ kg has moved relative to the fixed pulley.

(i)



5

5

5

15

4. cont.

(ii)

$T - 2T_1 - mg = mf$ 5

$5mg - T = 5mf$ 5

$T_1 - mg = m(a + f)$ 5

$2mg - T_1 = 2m(a - f)$ 5

$4mg - 2T_1 = 6mf$ (add eqs 1 and 2)

$3T_1 - 4mg = 4mf$ (eliminate a from eqs 3 and 4)

$\Rightarrow \quad f = \dfrac{2g}{13}$ and $a = \dfrac{5g}{13}$ 5

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(iii)

2m mass :

$s = ut + \tfrac{1}{2}(a - f)t^2$

$1 = 0 + \dfrac{3g}{26}t^2 \quad \Rightarrow \quad t^2 = \dfrac{26}{3g}$ 5

5m mass :

$s = ut + \tfrac{1}{2}(f)t^2$

$s = 0 + \dfrac{1}{2}\left\{\dfrac{2g}{13}\right\}\left\{\dfrac{26}{3g}\right\} \quad \Rightarrow s = \dfrac{2}{3}$ m. 5

10

5. **(a)** A uniform smooth sphere of mass 2 kg and moving with speed u m/s collides with another smooth sphere of mass 3 kg which is at rest. The velocity of the sphere of mass 2 kg before impact makes an angle of 45° with the line of centres at impact. The coefficient of restitution between the spheres is e.

(i) Find, in terms of e and u, the speed of each sphere after the collision.

(ii) If the sphere of mass 2 kg makes an angle $\tan^{-1} 10$ with the line of centres after impact, find e.

PCM $\quad 2\left(\frac{u}{\sqrt{2}}\right)+3(0) = 2v_1 + 3v_2$ 5

NEL $\quad v_1 - v_2 = -e\left(\frac{u}{\sqrt{2}}\right)$ 5

$\Rightarrow \quad v_1 = \frac{u}{5\sqrt{2}}(2-3e) \quad$ and $\quad v_2 = \frac{u}{5\sqrt{2}}(2+2e)$ 5

Speed of first sphere $= \sqrt{\left\{\frac{u}{5\sqrt{2}}(2-3e)\right\}^2 + \left\{\frac{u}{\sqrt{2}}\right\}^2}$

Speed of second sphere $= \frac{u}{5\sqrt{2}}(2+2e)$

$$10 = \frac{\frac{u}{\sqrt{2}}}{\frac{u}{5\sqrt{2}}(2-3e)} \quad \text{or} \quad 10 = \frac{\frac{u}{\sqrt{2}}}{-\frac{u}{5\sqrt{2}}(2-3e)}$$ 5

$\Rightarrow \quad e = \frac{1}{2} \quad$ or $\quad e = \frac{5}{6}$ 5

25

(b) Two identical smooth spheres, each of mass m and moving in the same direction collide directly. The coefficient of restitution between the spheres is e.
If u is the magnitude of the relative velocity between the spheres before impact, show that

(i) each sphere receives an impulse of magnitude $\frac{1}{2}mu(1+e)$

(ii) the loss in the total kinetic energy of the two spheres due to the impact is $\frac{1}{4}mu^2(1-e^2)$.

Relative to second sphere :

PCM $\quad mu + m(0) = mv_1 + mv_2$ 5

NEL $\quad v_1 - v_2 = -e(u-0)$ 5

$$v_1 + v_2 = u$$
$$v_1 - v_2 = -eu$$

$$v_1 = \tfrac{1}{2}u(1-e)$$
$$v_2 = \tfrac{1}{2}u(1+e)$$ 5

$$\left\{\begin{array}{rl} \text{Impulse} & = |mv_2 - mu_2| \\ & = \tfrac{1}{2}mu(1+e) \end{array}\right\}$$

OR

$$\left\{\begin{array}{rl} \text{Impulse} & = |mv_1 - mu_1| \\ & = |\tfrac{1}{2}mu(1-e) - mu| \\ & = |-\tfrac{1}{2}mu(1+e)| \\ & = \tfrac{1}{2}mu(1+e) \end{array}\right\}$$ 5

$$\begin{array}{rl} \text{Loss in KE} & = \tfrac{1}{2}m\{u_1^2 + u_2^2 - v_1^2 - v_2^2\} \\ & = \tfrac{1}{2}m\{u^2 + 0 - \{\tfrac{1}{2}u(1-e)\}^2 - \{\tfrac{1}{2}u(1+e)\}^2\} \\ & = \tfrac{1}{2}m\{u^2 - \tfrac{1}{4}u^2\{1 - 2e + e^2 + 1 + 2e + e^2\}\} \\ & = \tfrac{1}{4}mu^2[1-e^2] \end{array}$$ 5

25

6. **(a)** A particle moving with simple harmonic motion has speeds of 5 cm/s and 2 cm/s when it is at points 3 cm and 4 cm, respectively, from the centre of the motion.

(i) Find the amplitude and the period of the motion.

(ii) Find the maximum speed of the particle.

(i)

$$v = \omega\sqrt{a^2 - x^2}$$

$$5 = \omega\sqrt{a^2 - 3^2}$$ 5

$$2 = \omega\sqrt{a^2 - 4^2}$$ 5

$$\frac{5}{2} = \frac{\sqrt{a^2 - 9}}{\sqrt{a^2 - 16}}$$

$$\Rightarrow a = 4.16 \quad \text{or} \quad \sqrt{\frac{52}{3}}$$

$$\Rightarrow \omega = 1.732 \ \text{ or } \ \sqrt{3}$$

$$\Rightarrow \ \mathrm{T} = \frac{2\pi}{\omega} = 3.63 \ \text{ or } \ \frac{2\pi}{\sqrt{3}}$$ 5

(ii)

$$\begin{aligned} v_{\max} &= \omega a \\ &= (1.732)(4.16) \\ &= 7.2 \ \text{ or } \ \sqrt{52} \ \text{ or } \ 2\sqrt{13} \ \text{cm/s} \end{aligned}$$ 5

20

(b) A particle of mass m kg is suspended from a fixed point p by a light elastic string of natural length l and elastic constant $\frac{4mg}{l}$.

(i) Find the distance of the equilibrium position from the point p, in terms of l.

(ii) The particle is pulled down until it is a distance $\frac{7l}{4}$ vertically below p and is then released from rest.

Find the time taken, in terms of l, for the string to go slack.

(i)

$$\text{ke} = \text{mg}$$ 5

$$\left\{\frac{4\text{mg}}{\ell}\right\}\text{e} = \text{mg} \qquad \Rightarrow e = \tfrac{1}{4}\ell$$

$$\Rightarrow \text{distance from p} = \ell + \tfrac{1}{4}\ell = \tfrac{5}{4}\ell$$ 5

(ii)

$$\text{Force} = \text{mg} - \text{T}$$

$$= \text{mg} - \left\{\frac{4\text{mg}}{\ell}\right\}(\text{e}+\text{x}) \quad = -\left\{\frac{4\text{mg}}{\ell}\right\}\text{x}$$

$$\text{acceleration} = -\left\{\frac{4\text{g}}{\ell}\right\}\text{x}$$

$$\Rightarrow \omega = \sqrt{\frac{4g}{\ell}}$$ 5

$$\text{amplitude} = \frac{7\ell}{4} - \frac{5\ell}{4} = \tfrac{1}{2}\ell$$ 5

$$x = a\sin\omega t_1$$

$$\tfrac{1}{4}\ell = \tfrac{1}{2}\ell\sin\left\{\sqrt{\frac{4g}{\ell}}(t_1)\right\}$$

$$\Rightarrow t_1 = \frac{\pi}{6}\sqrt{\frac{\ell}{4g}}$$ 5

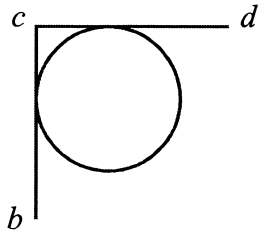
$$\text{Time taken} = \tfrac{1}{4}\text{Period} + \text{t}_1$$

$$= \frac{\pi}{2}\sqrt{\frac{\ell}{4g}} + \frac{\pi}{6}\sqrt{\frac{\ell}{4g}}$$

$$= \frac{4\pi}{6}\sqrt{\frac{\ell}{4g}} \text{ or } \frac{\pi}{3}\sqrt{\frac{\ell}{g}}$$ 5

30

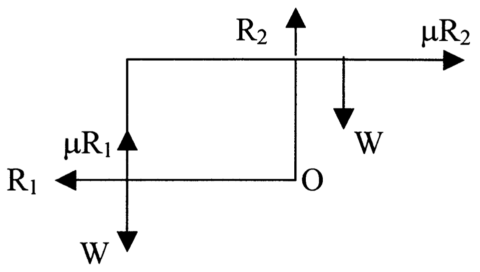
7. **(a)** Two identical uniform rods, [*bc*] and [*cd*], each of weight W and each of length 4*l*, are rigidly connected at *c* so that $|\angle bcd| = 90°$.
The rods rest in limiting equilibrium (that is, Just on the point of slipping) in contact with a fixed circular peg of radius *a* with rod [*cd*] horizontal and rod [bc] vertical and where $l < a < 2l$.
The coefficient of friction between each rod and the peg is μ where $\mu < 1$.



(i) Draw a diagram showing all the forces acting on the rods.

(ii) Show that $a = \dfrac{\ell(1+\mu^2)}{1-\mu}$

(i)



5

5

(ii)

$$R_1 = \mu R_2 \qquad \left\{R_1 = \frac{2\mu W}{1+\mu^2}\right\}$$ 5

$$R_2 + \mu R_1 = 2W \qquad \left\{R_2 = \frac{2W}{1+\mu^2}\right\}$$ 5

Moments about O :

$$W(a) = \mu R_1\,(a) + \mu R_2\,(a) + W(2\ell - a)$$ 5

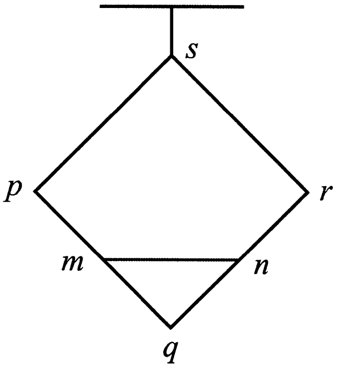
$$W(a) = \mu\left\{\frac{2\mu W}{1+\mu^2}\right\}(a) + \mu\left\{\frac{2W}{1+\mu^2}\right\}(a) + W(2\ell - a)$$

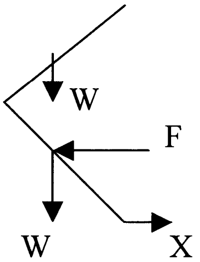
$$\Rightarrow \qquad a = \frac{\ell(1+\mu^2)}{1-\mu}$$ 5

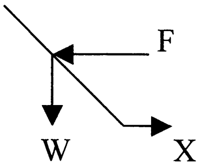
30

(b) Four identical uniform rods, each of weight W, are freely jointed at their ends to form a square $pqrs$. The square is suspended from s and is held in the form of a square by a light rod, $[mn]$, joining the midpoints of the rods $[pq]$ and $[qr]$.

Calculate, in terms of W, the force in the light rod $[mn]$.







Consider spq : take moments about s

$F(3) = X(4) + W(1) + W(1)$

Consider pq : take moments about p

$X(2) = F(1) + W(1)$

$\Rightarrow \quad F = 4W$

5

5

5

5

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8. **(a)** Prove that the moment of inertia of a uniform disc, of mass *m* and radius *r*, about an axis through its centre perpendicular to its plane is $\frac{1}{2}mr^2$.

Let M = mass per unit area

mass of element = $M\{2\pi\,x.dx\}$

moment of inertia of the element = $M\{2\pi\,x.dx\}x^2$ 5

moment of inertia of the disc $= 2\pi M \int_0^r x^3\,dx$ 5

$= 2\pi M\left[\frac{x^4}{4}\right]_0^r$ 5

$= \frac{1}{2}\pi M r^4$

$= \frac{1}{2}m r^2$ 5 20

(b) State the Parallel and Perpendicular Axes Theorems.

Hence, or otherwise, find the moment of inertia of a uniform disc of mass *m* and radius *r* about an axis tangential to its circumference and lying in the plane of the disc.

The disc may rotate smoothly about a fixed horizontal axis tangential to its circumference and lying in the plane of the disc. The disc is held in the horizontal plane and then released from rest.

Find the angular speed of the disc when it has rotated through an angle ϑ, in terms of *r* and ϑ.

Find, in terms of *r*, the maximum angular speed in the subsequent motion.

Perpendicular Axis Theorem:
If *X, Y* and Z are mutually perpendicular axes through a point in a lamina, with *X* and *Y* lying in the plane of the lamina, then $I_Z = I_X + I_Y$, where these represent the moments of inertia about the respective axes. 5

Parallel Axis Theorem:
If I_C is the moment of inertia of a rigid body of mass *m* about an axis through its centre of gravity, I is the moment of inertia of the body about a parallel axis and *d* is the distance between these two axes, then $I = I_C + md^2$. 5

$$I = \tfrac{1}{4}mr^2 + mr^2 = \tfrac{5}{4}mr^2$$ 5

$$\tfrac{1}{2}\{I\}\omega^2 = mgh$$

$$\tfrac{1}{2}\left\{\tfrac{5}{4}mr^2\right\}\omega^2 = mgr\sin\vartheta$$ 5

$$\omega = \sqrt{\frac{8g\sin\vartheta}{5r}}$$ 5

$$\omega_{\max} = \sqrt{\frac{8g}{5r}} \quad \text{i.e. when } \vartheta = \tfrac{\pi}{2}$$ 5

30

9. **(a)** A compound object made up of two bodies of relative densities 1.5 and 2.3 weighs 9.408 N in water and 4.704 N in a liquid of relative density 1.3.

Find the volume of each body in the compound object.

Water:

$B = (V_1 + V_2)1000g$ 5

$\{1500V_1g + 2300V_2g\} - \{(V_1 + V_2)1000g\} = 9.408$ 5

$500V_1g + 1300V_2g = 9.408$

$500V_1 + 1300V_2 = 0.96$

Liquid:

$B = (V_1 + V_2)1300g$ 5

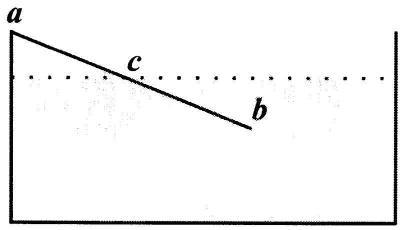
$\{1500V_1g + 2300V_2g\} - \{(V_1 + V_2)1300g\} = 4.704$

$200V_1g + 1000V_2g = 4.704$

$200V_1 + 1000V_2 = 0.48$

$V_1 = 14 \times 10^{-4}$ and $V_2 = 2 \times 10^{-4}$ 5 20

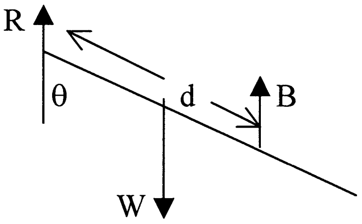
(b) A uniform rod, $[ab]$, is of length $2l$. The end a of the rod is hinged smoothly at the edge of a tank. The rod is in an inclined position with part $[cb]$ in the uniform liquid in the tank. The density of the rod is ρ and the density of the liquid is σ.
The rod is at rest.



Show that the length of the immersed part, $[cb]$, of the rod is

$$2l\left(1-\sqrt{1-\frac{\rho}{\sigma}}\right).$$

(b)



5

Let $[\text{cb}] = x$

$$\text{d} = 2\ell - \tfrac{1}{2}x$$ 5

Take moments about a

$$B\{2\ell - \tfrac{1}{2}x\}\cos\theta = W\{\ell\cos\theta\}$$ 5

$$B = \frac{W_l s_L}{s} = \frac{x}{2\ell}W\frac{\sigma}{\rho} \quad \text{or} \quad B = \sigma\left(\frac{\text{xV}}{2\ell}\right)g$$ 5

$$\{\,\text{W} = \rho\text{Vg}\}$$

$$\sigma x^2 - 4\ell\sigma x + 4\ell^2\rho = 0$$ 5

$$x = 2\ell\left\{1-\sqrt{1-{}^{\rho}\!/_{\sigma}}\right\}$$ 5

30

10. **(a)** Find $\dfrac{d}{dx}\left(\dfrac{y}{x}\right)$.

Hence, or otherwise, solve the differential equation

$$\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}=\frac{1}{x}$$

given that $y = 1$ when $x = 1$.

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{y}{x}\right)=\frac{x\frac{dy}{dx}-y}{x^2}$$

$$=\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}$$

5

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{y}{x}\right)=\frac{1}{x} \quad or \quad \int \mathrm{d}\left(\frac{y}{x}\right)=\int\frac{\mathrm{dx}}{\mathrm{x}}$$

$$\frac{y}{x}=\ln x+\mathrm{C}$$

5

$$y=1, x=1 \Rightarrow \mathrm{C}=1$$

$$y=x(\ln x+1)$$

5 15

(b) A car of mass m kg is travelling along a level road. The resistance to motion is mkv^2 N, where v m/s is the speed. When the car is travelling at 14 m/s, the engine cuts out. Ten seconds after the engine cuts out, the speed of the car is 7 m/s.

(i) Show that $k=\dfrac{1}{140}$.

(ii) The car travels a distance of s metres in the first T seconds after the engine cuts out.
Show that

$$s=140\ln\left(1+\frac{T}{10}\right).$$

(i)

$$\text{Force} = -mkv^2$$

5

$$m\frac{dv}{dt}=-mkv^2$$

$$-\int\frac{dv}{v^2}=k\int dt$$

5

$$\left[\frac{1}{v}\right]_{14}^{7}=k[\mathrm{t}]_0^{10}$$

5

$$\frac{1}{7}-\frac{1}{14}=k(10) \Rightarrow k=\frac{1}{140}$$

5 20

10 (b) cont.

(ii)

$$v\frac{dv}{dx} = -kv^2$$ 5

$$\int\frac{dv}{v} = -k\int dx$$

$$[\ln V]_{14}^{V} = -ks$$

$$\ln\left(\frac{V}{14}\right) = -ks$$ 5

$$V = 14e^{-ks}$$

$$\frac{ds}{dt} = 14e^{-ks}$$

$$\int e^{ks}ds = 14\int dt$$

$$\left[\frac{1}{k}e^{ks}\right]_0^s = 14T$$

$$140\{e^{ks} - 1\} = 14T$$

$$s = 140\ln\left(1+\frac{T}{10}\right)$$ 5 15

OR

$$\left[\frac{1}{v}\right]_{14}^{v} = k[\text{t}]_0^t$$ 5

$$\frac{1}{v} - \frac{1}{14} = \frac{1}{140}\text{t}$$

$$\Rightarrow\ v = \frac{140}{10+\text{t}}$$ 5

$$\frac{\text{ds}}{\text{dt}} = \frac{140}{10+\text{t}}$$

$$s = 140[\ln(10+t)]_0^T$$

$$= 140\{\ln(10+T) - \ln(10)\}$$

$$= 140\ln\left\{1+\frac{T}{10}\right\}$$ 5 15

Some alternative solutions

1 (a) second part:

$s_1 = 800$ or $s_2 = 400$	
$v^2 = u^2 + 2as$	
$v^2 = 0 + 2(1)(800)$ or $0 = v^2 + 2(-2)(400)$	
$v = 40$	5
$\frac{1}{2}T(40) = 1200 \quad \Rightarrow \quad T = 60$	5

$t_1 : t_2 = 2:1$ or $t_1 = 2t_2$	
$\frac{v}{t_1} = 1 \quad \Rightarrow \quad t_1 = v$ or $t_2 = \frac{1}{2}v$	5
$1200 = \frac{1}{2}(t_1 + t_2)v$ *or* $1200 = \frac{1}{2}(3t_2)(2t_2)$	
$t_1 = 40$ and $t_2 = 20$	
$\Rightarrow \quad T = 60$	5

1 (b)

$v_1 = u - g(T + t_1)$ or $v_1 = u - gt_2$	5
$v_2 = u - gt_1$ or $v_2 = u - g(t_2 - T)$	5
$v_1 = -v_2$ *or* $v_2 = -v_1$	5
$u - g(T + t_1) = -u + gt_1$ *or* $u - g(t_2 - T) = -u + gt_2$	
$2u - gT = 2gt_1$ *or* $2u + gT = 2gt_2$	
$T + t_1 = \frac{\mathrm{u}}{\mathrm{g}} + \frac{T}{2}$ or $\mathrm{t}_2 = \frac{\mathrm{u}}{\mathrm{g}} + \frac{T}{2}$	5
height $= \mathrm{s}_1 = \mathrm{s}_2$	
$= u\left\{\frac{\mathrm{u}}{\mathrm{g}} \pm \frac{T}{2}\right\} - \frac{1}{2}\mathrm{g}\left\{\frac{\mathrm{u}}{\mathrm{g}} \pm \frac{T}{2}\right\}^2$	
$= \frac{4\mathrm{u}^2 - g^2T^2}{8g}$	5

2 (b)

$160 = \left| u\,\vec{j} - V_w \right|$ 5

$160 = \left| -16\sqrt{2}\,\vec{i} + \{u - 16\sqrt{2}\}\vec{j} \right|$ 5

$25600 = 1024 + u^2 - 32\sqrt{2}u$

$\Rightarrow u = 181.02$ 5

$160 = \left| -u\,\vec{j} - V_w \right|$

$160 = \left| -16\sqrt{2}\,\vec{i} - \{u + 16\sqrt{2}\}\vec{j} \right|$

$25600 = 1024 + u^2 + 32\sqrt{2}u$

$\Rightarrow u = 135.76$ 5

$\text{time from } p \text{ to } q = \dfrac{135.76}{181.02 + 135.76} \times 5 = 2.14$

$\text{time from } q \text{ to } p = \dfrac{181.02}{181.02 + 135.76} \times 5 = 2.86$

$d = 2.14 \times 181.02 \text{ or } 2.86 \times 135.76$

$\Rightarrow \text{d} = 387.4 \text{ or } 388.3 \text{ km}$ 5 25

5 (b)

$$\begin{array}{ll} \text{PCM} & \mathrm{mu_1 + mu_2 = mv_1 + mv_2} \\ \text{NEL} & \mathrm{v_1 - v_2 = -e(u_1 - u_2)} \end{array}$$

5

5

$$\mathrm{v_1} = \tfrac{1}{2}\mathrm{u_1}(1-e) + \tfrac{1}{2}\mathrm{u_2}(1+e)$$
$$\mathrm{v_2} = \tfrac{1}{2}\mathrm{u_1}(1+e) + \tfrac{1}{2}\mathrm{u_2}(1-e)$$

5

$$\left\{\begin{aligned} \text{Impulse} &= \left|\mathrm{mv_2 - mu_2}\right| \\ &= \tfrac{1}{2}\mathrm{mu_1}(1+e) + \tfrac{1}{2}\mathrm{mu_2}(1-e) - \mathrm{mu_2} \\ &= \tfrac{1}{2}\mathrm{mu_1}(1+e) - \tfrac{1}{2}\mathrm{mu_2}(1+e) \\ &= \tfrac{1}{2}\mathrm{m(u_1 - u_2)}(1+e) \\ &= \tfrac{1}{2}\mathrm{mu}(1+e) \end{aligned}\right\}$$

OR

$$\left\{\begin{aligned} \text{Impulse} &= \left|\mathrm{mv_1 - mu_1}\right| \\ &= \left|\tfrac{1}{2}\mathrm{mu_1}(1-e) + \tfrac{1}{2}\mathrm{mu_2}(1+e) - \mathrm{mu_1}\right| \\ &= \left|-\tfrac{1}{2}\mathrm{mu_1}(1+e) + \tfrac{1}{2}\mathrm{mu_2}(1+e)\right| \\ &= \left|-\tfrac{1}{2}\mathrm{m(u_1 - u_2)}(1+e)\right| \\ &= \tfrac{1}{2}\mathrm{mu}(1+e) \end{aligned}\right\}$$

5

$$\begin{aligned} \text{Loss in KE} &= \tfrac{1}{2}\mathrm{m}\left\{\mathrm{u_1^2 + u_2^2 - v_1^2 - v_2^2}\right\} \\ &= \tfrac{1}{2}\mathrm{m}\left\{\begin{array}{l} \mathrm{u_1^2 + u_2^2} - \left(\tfrac{1}{4}u_1^2\left[1-2e+e^2\right] + \tfrac{1}{2}u_1u_2\left[1-e^2\right] + \tfrac{1}{4}u_2^2\left[1+2e+e^2\right]\right) \\ -\left(\tfrac{1}{4}u_1^2\left[1+2e+e^2\right] + \tfrac{1}{2}u_1u_2\left[1-e^2\right] + \tfrac{1}{4}u_2^2\left[1-2e+e^2\right]\right) \end{array}\right\} \\ &= \tfrac{1}{2}\mathrm{m}\left\{\mathrm{u_1^2 + u_2^2} - \left(\tfrac{1}{4}u_1^2\left[2+2e^2\right] + u_1u_2\left[1-e^2\right] + \tfrac{1}{4}u_2^2\left[2+2e^2\right]\right)\right\} \\ &= \tfrac{1}{2}\mathrm{m}\left\{\tfrac{1}{2}\mathrm{u_1^2} + \tfrac{1}{2}\mathrm{u_2^2} - \left(\tfrac{1}{2}u_1^2e^2 + u_1u_2\left[1-e^2\right] + \tfrac{1}{2}u_2^2e^2\right)\right\} \\ &= \tfrac{1}{2}\mathrm{m}\left\{\tfrac{1}{2}\mathrm{u_1^2}\left[1-e^2\right] + \tfrac{1}{2}\mathrm{u_2^2}\left[1-e^2\right] - u_1u_2\left[1-e^2\right]\right\} \\ &= \tfrac{1}{4}\mathrm{m}\left[1-e^2\right]\left\{\mathrm{u_1^2 + u_2^2} - 2u_1u_2\right\} \\ &= \tfrac{1}{4}\mathrm{m}\left[1-e^2\right]\left\{\mathrm{u_1} - u_2\right\}^2 \\ &= \tfrac{1}{4}\mathrm{mu^2}\left[1-e^2\right] \end{aligned}$$

5

25

9 (a)

Water:

$B_1 = \frac{W_1}{1.5}$ and $B_2 = \frac{W_2}{2.3}$ 5

$W_1 + W_2 - \frac{W_1}{1.5} - \frac{W_2}{2.3} = 9.408$ 5

Liquid:

$B_1 = \frac{W_1(1.3)}{1.5}$ and $B_2 = \frac{W_2(1.3)}{2.3}$ 5

$W_1 + W_2 - \frac{W_1(1.3)}{1.5} - \frac{W_2(1.3)}{2.3} = 4.704$

$\Rightarrow W_1 = 20.58$ and $W_2 = 4.51$

$\Rightarrow 1500\{V_1\}g = 20.58$ and $2300\{V_2\}g = 4.51$

$\Rightarrow \mathrm{V}_1 = 14 \times 10^{-4}$ and $\mathrm{V}_2 = 2 \times 10^{-4}$ 5 20

s	V	sV
1.5	A	$1.5A$
2.3	B	$2.3B$
	$A+B$	$1.5A+2.3B$

Water:

$B = \frac{W}{\bar{s}}$ and $\bar{s} = \frac{sV}{V} = \frac{1.5A + 2.3B}{A + B}$ 5

$W - \frac{W}{\bar{s}} = 9.408$ 5

Liquid:

$B = \frac{W(1.3)}{\bar{s}}$ and $\bar{s} = \frac{sV}{V} = \frac{1.5A + 2.3B}{A + B}$ 5

$W - \frac{W(1.3)}{\bar{s}} = 4.704$

$\Rightarrow W = 25.088$ and $\bar{s} = 1.6$

$\Rightarrow W = 1000(\bar{s})g\{A + B\}$ and $A = 7B$

$\Rightarrow \mathrm{A} = 14 \times 10^{-4}$ and $\mathrm{B} = 2 \times 10^{-4}$ 5 20

10.(a) second part

$$\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}=\frac{1}{x}$$

$$\frac{dy}{dx}-\frac{y}{x}=1$$ 5

Let $y = ux \Rightarrow \dfrac{dy}{dx}=u+x\dfrac{du}{dx}$

$$\Rightarrow x\frac{du}{dx}=1$$

$$\int \mathrm{d}u=\int\frac{\mathrm{dx}}{\mathrm{x}}$$

$$\frac{y}{x}=\ln x+\mathrm{C}$$

$$y=1, x=1 \Rightarrow \mathrm{C}=1$$

$$y=x(\ln x+1)$$ 5